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Fluid membranes in hydrodynamic flow fields: Formalism and an application to fluctuating quasispherical vesicles in shear flow

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Abstract. The dynamics of a single fluid bilayer membrane in an external hydrodynamic flow field is considered. The deterministic equation of motion for the configuration is derived taking into account both viscous dissipation in the surrounding liquid and local incompressibility of the membrane. For quasi-spherical vesicles in shear flow, thermal fluctuations can be incorporated in a Langevin-type equation of motion for the deformation amplitudes. The solution to this equation shows an overdamped oscillatory approach to a stationary tanktreading shape. Inclination angle and ellipticity of the contour are determined as a function of excess area and shear rate. Comparisons to numerical results and experiments are discussed.

PACS. 47.15.Gf Low-Reynolds-number (creeping) flows – 68.15.+e Liquid thin films – 82.70.-y Disperse systems

1 Introduction

The equilibrium behavior of fluid vesicles made up of a single lipid bilayer membrane has been studied in great detail and is by now well understood [1]. These equilibrium phenomena comprise the occurrence of a multitude of vesicle shapes and their thermal fluctuations. If external parameters such as temperature or osmotic conditions are changed, such a shape may become unstable and settle into the next dynamically accessible minimum of bending energy. The dynamics of such a shape change, however, is a non-equilibrium process. In general, only very little is known systematically and quantitatively about this and other non-equilibrium phenomena of fluid vesicles.

A rough classification of non-equilibrium phenomena should distinguish between two types. First, there is relaxation into a new equilibrium after a parameter change. This class comprises the above mentioned decay of a metastable shape such, *e.g.*, as the budding process [2,3]. The best studied case of this class is the spectacular pearling instability of cylindrical vesicles, which develops upon action of optical tweezers [4].

The second class of nonequilibrium behavior refers to genuine non-equilibrium states induced by external fields such as hydrodynamic flow fields. The behavior of vesicles under such conditions has only recently been started to be explored theoretically [5–7] and in experiment [8]. We developed a numerical code which allows to follow the shape evolution of vesicle in shear flow [5]. As a main result, we found that the shape finally settles into a non-axisymmetric stationary ellipsoidal shape around which the membrane exhibits tanktreading motion. Quantitative predictions include the inclination angle and the tank-treading frequency as a function of reduced volume and shear rate. This work left aside two interesting aspects. First, the regime of very small shear rates was difficult to reach within this numerical approach because of long relaxation times. Even though the numerical data seemed to indicate that shear is a singular perturbation, no clear assessment of this regime was possible. Secondly, fluctuations, which would be particularly important at small shear rate, were not included into the numerical algorithm.

The purpose of this paper is twofold. First, we present somewhat more explicitly the underlying continuum theory for the dynamical evolution of an incompressible fluid membrane in an external flow field thereby generalizing previous work on dynamics of membranes in quiescent fluids [9] and within a Rouse model [10]. The resulting equations are strongly non-linear and must, in general, be solved numerically. Analytical progress, however, is possible for so-called quasispherical vesicles [11–17] for which deviations from a sphere remain small throughout the parameter space. Focussing on such vesicles, it becomes possible to include the effect of thermal fluctuations which is the second objective of this paper. A main effect of thermal fluctuations in fluid membranes is that they store or "hide" area in the suboptical range [18]. The present work addresses the issue of how external flow pulls out hidden area from thermal fluctuations. Apart from its fundamental significance such a study is motivated also by the so far

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only published experiments on the behavior of vesicles in shear flow [8]. In this experiment initially spherical vesicles elongate under shear thereby pulling out area from thermal fluctuations. The theory developed here should be used to analyze such experiments.

For a somewhat broader perspective, it may be important to point out that there exists a vast literature on the behavior of a "soft sphere" in shear flow ranging from liquid droplets [19,20] to elastic capsules [21–23] or inert ellipsoids [24–26] modeling red-blood-cells [27] or synthetic microcapsules [28]. Many of these systems show ellipsoidal deformations with revolving or tanktreading motion. The details of deformation, inclination angle and tanktreading frequency, however, depend crucially on both the constitutive relation for the elasticity of the "interface" of the soft sphere and the dissipative mechanisms involved. In our case, the membrane is determined by bending elasticity and local incompressibility. The main dissipation occurs in the surrounding fluid.

How thermal fluctuations are affected by shear flow seems not to have been touched upon in these studies of soft spheres. For the topologically much simpler case of on average planar membranes in a dilute lamellar phase, the layer spacing is predicted to decrease with increasing shear due to the suppression of fluctuations [29,30].

This paper consists of five sections, of which this is the first. Section 2 discusses the physical model underlying this work and derives the fundamental equation of motion for an incompressible membrane in external flow. In Section 3, we specialize to quasi-spherical vesicles in linear flow. In Section 4, we discuss in detail shear flow for which we solve the equations explicitly and compare to available numerical and experimental work. A summarizing discussion is presented in Section 5.

2 Formalism

2.1 Physical model

The starting point for the dynamics of a fluid membrane is the bending energy [31]

$$F_\kappa \equiv \frac{\kappa}{2}\int dA(2H - C_0)^2. \tag{1}$$

Here, κ is the bending rigidity, H is the local mean curvature and C_0 a spontaneous curvature in the case of intrinsically asymmetric monolayers or an asymmetric liquid environment. We neglect Gaussian curvature energy which would enter only through boundary values, since we focus on the dynamics of a closed membrane. For simplicity, we also neglect the fact that for closed membranes a second energy term becomes relevant which takes into account that the bilayer consists of two tightly coupled monolayers which do not exchange molecules [32].

The bending energy is the driving force for dynamical changes. Dynamics in the micron world of vesicles is overdamped, *i.e.* inertial effects can safely be ignored as can easily be checked *a posteriori* by calculating the corresponding Reynolds number [33]. Dissipation takes place both in the surrounding liquid and in the membrane, in principle. For giant vesicles of micron size, the dominant dissipation is viscous dissipation in the embedding fluid [34,35]. Therefore, a full treatment of the hydrodynamics of this fluid is mandatory for a faithful description of the dynamics of membranes.

Dissipation in the membrane can be classified into three phenomena: Drag between the two monolayers [36], shear viscosity within each layer and permeation through the membrane. Calculation of the relaxation spectra of bending fluctuations involving the first two mechanisms show that on scales of microns and larger, hydrodynamic dissipation is dominant [37]. In the submicron range, friction between the layers becomes relevant. On even smaller scales of several nanometer, shear viscosity within each layer should be included. Finally, permeation through the membrane seems to be irrelevant on all length-scales with the possible exception of membranes in the vicinity of a substrate [38].

Based on this hierarchy of dissipative mechanisms, we will include only hydrodynamic dissipation and model the membrane as impermeable for liquid. As a consequence, the normal velocity of the fluid at the membrane pushes along the membrane and leads to a shape change. Tangential motion of the fluid along the membrane induces lipid flow within the membrane since we employ non-slip boundary conditions between fluid and membrane. One could allow for some slip with a phenomenological friction coefficient. In the absence of any evidence for such a phenomenon, however, we choose for simplicity the no-slip condition used so far. Finally, we require that the membrane remains locally incompressible.

2.2 Elementary differential geometry of the membrane

The instantaneous membrane configuration $\mathbf{R}(s_1, s_2)$, parametrized by internal coordinates (s_1, s_2), is embedded in the three dimensional space. This space will be parametrized by Cartesian (x, y, z) or spherical (r, θ, ϕ) coordinates as $\mathbf{r} = x_\alpha \mathbf{e}_\alpha = r\mathbf{e}_r$ where $\alpha = x, y, z$. Summation over double indices is implied throughout the paper. There are two tangential vectors

$$\mathbf{R}_i \equiv \partial_i \mathbf{R}(s_1, s_2) \quad \text{for} \quad i = s_1, s_2, \tag{2}$$

from which one obtains the metric tensor

$$g_{ij} \equiv \mathbf{R}_i \cdot \mathbf{R}_j. \tag{3}$$

Its determinant, $g \equiv \det(g_{ij})$, yields the area element

$$dA = \sqrt{g}\, ds_1 ds_2. \tag{4}$$

The normal vector, $\mathbf{n}(s_1, s_2)$, is given by

$$\mathbf{n} = \frac{\mathbf{R}_1 \times \mathbf{R}_2}{|\mathbf{R}_1 \times \mathbf{R}_2|}. \tag{5}$$

Finally, the mean and the Gaussian curvature follow from the curvature tensor

$$h_{ij} \equiv (\partial_i \partial_j \mathbf{R}) \cdot \mathbf{n} \tag{6}$$

as

$$H \equiv \frac{1}{2}\, h^i{}_i \tag{7}$$

and

$$K \equiv \det(h^i{}_j), \tag{8}$$

where $h^i_j \equiv g^{ik} h_{kj}$ and g^{ij} are the matrix elements of the matrix inverse of (g_{ij}). Following the convention used in differential geometry, a sphere with the usual spherical coordinates $(s_1 = \theta, s_2 = \phi)$ has $H < 0$. A membrane configuration has bending energy F_κ given in (1).

In order to ensure local incompressibility of the membrane, we will need a local Lagrange multiplier $\Sigma(s_1, s_2)$ which we call local surface tension. It will be determined self-consistently below. The total "energy" thus becomes

$$F \equiv F_\kappa + \int ds_1 ds_2 \sqrt{g}\ \Sigma(s_1, s_2). \tag{9}$$

2.3 Equation of motion

Any non-equilibrium membrane configuration exerts a local three dimensional force density

$$\mathbf{f}(\mathbf{r}) \equiv -\int ds_1 ds_2 \sqrt{g} \left(\frac{1}{\sqrt{g}} \frac{\delta F}{\delta \mathbf{R}} \right) \delta(\mathbf{r} - \mathbf{R}(s_1, s_2)) \tag{10}$$

onto the surrounding fluid.

The variational derivative entering the force density reads explicitly

$$\left(\frac{1}{\sqrt{g}} \frac{\delta F}{\delta \mathbf{R}} \right) = (-2\Sigma H + \kappa[(2H - C_0)(2H^2 - 2K + C_0 H) + 2\Delta H])\mathbf{n} - g^{ij}\mathbf{R}_i \partial_j \Sigma. \tag{11}$$

Here,

$$\Delta \equiv (1/\sqrt{g})\partial_i (g^{ij} \sqrt{g} \partial_j). \tag{12}$$

is the Laplace-Beltrami operator on the surface. The normal part of (11) is well-known from the stationarity condition of membrane configurations [39]. The tangential part arises from inhomogeneities in the surface tension which will be needed to ensure local incompressibility of the induced flow. This approach differs from the one used in reference [9] where a finite compressibility was introduced for renormalization purposes.

The surrounding liquid is incompressible

$$\nabla \mathbf{v} = 0 \tag{13}$$

and obeys the Stokes equations

$$\nabla p - \eta \nabla^2 \mathbf{v} = \mathbf{f}(\mathbf{r}). \tag{14}$$

A special solution of the inhomogeneous Stokes equation for the velocity field reads

$$\mathbf{v}^{ind}(\mathbf{r}) = \int d\mathbf{r}' \mathcal{O}(\mathbf{r}, \mathbf{r}') \mathbf{f}(\mathbf{r}'), \tag{15}$$

where the Oseen tensor $\mathcal{O}(\mathbf{r}, \mathbf{r}')$ has Cartesian matrix elements [40]

$$\mathcal{O}_{\alpha\beta}(\mathbf{r}, \mathbf{r}') \equiv \frac{1}{8\pi\eta |\mathbf{r} - \mathbf{r}'|} \left[\delta_{\alpha\beta} + \frac{(r_\alpha - r'_\alpha)(r_\beta - r'_\beta)}{|\mathbf{r} - \mathbf{r}'|^2} \right]. \tag{16}$$

Thus, the hydrodynamics mediates a long-range interaction ($\sim 1/|\mathbf{r} - \mathbf{r}'|$) through the velocity field. To this velocity field $\mathbf{v}^{ind}(\mathbf{r})$ induced by the presence of the membrane, we must add the externally applied flow field $\mathbf{v}^\infty(\mathbf{r})$ to obtain the total velocity field

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}^\infty(\mathbf{r}) + \mathbf{v}^{ind}(\mathbf{r}). \tag{17}$$

Such a simple superposition becomes possible since the Stokes equation is linear due to the absence of the convective term. The value of the total velocity field at the position of the membrane yields the equation of motion for the membrane configuration

$$\partial_t \mathbf{R}(s_1, s_2) = \mathbf{v}^\infty(\mathbf{R}(s_1, s_2)) + \mathbf{v}^{ind}(\mathbf{R}(s_1, s_2)), \tag{18}$$

since we employ no-slip boundary conditions at the membrane. This deterministic equation of motion includes both normal motion that signifies a shape change of the membrane and tangential motion that corresponds to lipid flow within the membrane. The so far unknown local tension $\Sigma(s_1, s_2)$ is determined by requiring that the membrane flow induced by this equation obeys local incompressibility. Thus, we must demand

$$\partial_t \sqrt{g} = 0, \tag{19}$$

which implies explicitly

$$g^{ij} \mathbf{R}_i \partial_j (\partial_t \mathbf{R}) = 0. \tag{20}$$

Upon insertion of the equation of motion (18) with (10, 15), this condition becomes a partial differential equation for the unknown tension $\Sigma(s_1, s_2)$.

Equations (18, 20) yield a deterministic evolution equation for a membrane configuration under the action of bending energy and hydrodynamics. For any given initial membrane configuration, the solution to these equations will run into the next dynamically accessible local minimum of bending energy. In general, these equations must be solved numerically. For vesicles in shear flow, this has been achieved recently [5].

3 Quasi-spherical vesicle in flow

In this section, we apply the general formalism to quasi-spherical vesicles [11–17] for which analytical progress becomes possible.

3.1 Expansion around the sphere

A quasispherical vesicle is characterized by its volume

$$V \equiv \frac{4\pi}{3}R^3, \tag{21}$$

which defines R, and its fixed area

$$A \equiv (4\pi + \Delta)R^2, \tag{22}$$

which defines the (dimensionless) excess area Δ.

The instantaneous vesicle shape $\mathbf{R}(\theta,\phi) \equiv R(\theta,\phi)\mathbf{e}_r$ can be parametrized by spherical harmonics

$$R(\theta,\phi) = R\left(1 + \sum_{l\geq 0}^{l_{max}} \sum_{m=-l}^{l} u_{l,m}\mathcal{Y}_{lm}(\theta,\phi)\right), \tag{23}$$

where $|m| \leq l$ and $u_{l,-m} = (-1)^m u_{l,m}^*$. The upper cutoff l_{max} is of order R/d where d is the membrane thickness. For a vesicle with $R \simeq 10{-}50\,\mu$m, l_{max} is of order 10^4. Since spontaneous curvature is irrelevant for the shape behaviour of quasi-spherical vesicles [17], we set for the rest of the paper $C_0 = 0$. Expanding the geometrical quantities as well as the bending energy around a sphere, one has [12,13,39]

$$F_\kappa/\kappa = 8\pi + \frac{1}{2}\sum_{l\geq 0}^{l_{max}} \sum_{m=-l}^{l} |u_{l,m}|^2(l+2)(l+1)l(l-1) + O(u_{l,m}^3), \tag{24}$$

$$A = R^2\left(4\pi\left(1+\frac{u_{0,0}}{\sqrt{4\pi}}\right)^2 + \sum_{l\geq 1}^{l_{max}} \sum_{m=-l}^{l} |u_{l,m}|^2 \times (1 + l(l+1)/2) + O(u_{l,m}^3)\right), \tag{25}$$

and

$$V = R^3\left(\frac{4\pi}{3}\left(1+\frac{u_{0,0}}{\sqrt{4\pi}}\right)^3 + \sum_{l\geq 1}^{l_{max}} \sum_{m=-l}^{l} |u_{l,m}|^2 + O(u_{l,m}^3)\right). \tag{26}$$

The volume constraint (21) fixes the amplitude $u_{0,0}$ as a function of the other amplitudes

$$u_{0,0} = -\sum_{l\geq 1}^{l_{max}} \sum_{m=-l}^{l} |u_{l,m}|^2/\sqrt{4\pi}, \tag{27}$$

where we truncate from now on the cubic terms. If this value is inserted into (25), the area constraint (22) becomes

$$\sum_{l\geq 1}^{l_{max}} \sum_{m=-l}^{l} |u_{l,m}|^2 \frac{(l+2)(l-1)}{2} = \Delta. \tag{28}$$

Since the ($l = 1$)-modes correspond to translations, which have to be omitted, from now on all sums start at $l = 2$. We will abbreviate

$$\sum_{l,m} \equiv \sum_{l\geq 2}^{l_{max}} \sum_{m=-l}^{l}. \tag{29}$$

We now add the global area constraint (28) with a Lagrangian multiplier

$$\Sigma \equiv \kappa\sigma/R^2 \tag{30}$$

to the quadratic part of the curvature energy. This leads to a quadratic expression for the total "energy" (9)

$$F = \frac{\kappa}{2}\sum_{l,m} E_l |u_{l,m}|^2, \tag{31}$$

with

$$E_l = (l+2)(l-1)[l(l+1) + \sigma]. \tag{32}$$

The instantaneous state of a vesicle must be characterized not only by the set of its deformations $u_{l,m}$ but also by its instantaneous surface tension

$$\Sigma(\theta,\phi) \equiv \frac{\kappa}{R^2}(\sigma + \sum_{l,m} \sigma_{l,m}\mathcal{Y}_{l,m}(\theta,\phi)). \tag{33}$$

The homogeneous value $\sigma = \sigma_{0,0}$, which will be called effective tension, has already been included into the energy (32).

3.2 Velocity field

The notion of a quasi-spherical vesicle implies that the deviations $u(\theta,\phi)$ from the spherical shape are small. Then stresses caused by bending moments and an inhomogeneous surface tension can be assumed to act on the sphere rather than on the deformed vesicles's surface. Likewise, all velocity fields will be evaluated and matched at the sphere. We retain this procedure for arbitrary flow strength even though it is strictly valid only for small external fields. As will be discussed in Section 4.4 below, while not modifying scaling laws, this approximation causes an error in numerical prefactors at large shear rates of at most 30%.

The instantaneous stress distribution caused by both the deformation and the inhomogeneous surface tension induces an irrotational velocity field $\mathbf{v}^{ind}(\mathbf{r})$ which we determine in the Appendix adapting the classical Lamb solution [33]. For this purpose, it is convenient to define another three-dimensional velocity field

$$\mathbf{V}^{ind}(\mathbf{r}) \equiv \mathbf{v}^{ind}(R,\theta,\phi) \tag{34}$$

which extends the velocity field on the sphere that corresponds to the vesicle formally to all space. $\mathbf{V}^{ind}(\mathbf{r})$ is

r-independent. This velocity field can then be characterized by its normal component

$$X^{ind} \equiv \sum_{l,m} X^{ind}_{l,m} \mathcal{Y}_{l,m}(\theta,\phi) \equiv \mathbf{V}^{ind}(R,\theta,\phi)\mathbf{e}_r \tag{35}$$

and the negative of its divergence

$$Y^{ind} \equiv \sum_{l,m} Y^{ind}_{l,m} \mathcal{Y}_{l,m}(\theta,\phi) \equiv -R\,\nabla\mathbf{V}^{ind}, \tag{36}$$

where $\mathcal{Y}_{l,m}(\theta,\phi)$ are the usual spherical harmonics. Although the full three-dimensional velocity field $\mathbf{v}^{ind}(\mathbf{r})$ is divergenceless, the derived quantity $\mathbf{V}^{ind}(\mathbf{r})$, in general, is not.

Normal and tangential stress balances on the vesicle surface determine $\mathbf{v}^{ind}(\mathbf{r})$ as a function of the displacement $u(\theta,\phi)$ and the surface tension $\sigma(\theta,\phi)$. As shown in the Appendix, we can determine the linear relationship between the expansion coefficients of the induced velocity on the sphere $\{X^{ind}_{l,m}, Y^{ind}_{l,m}\}$ and the sources of this field $\{u_{l,m},\sigma_{l,m}\}$. Replacing $\sigma_{l,m}$ by $X^{ind}_{l,m}$ and $Y^{ind}_{l,m}$, one finally gets

$$X^{ind}_{l,m} = -(\kappa/\eta R^2)\;\Gamma_l E_l u_{l,m} - B_l Y^{ind}_{l,m}, \tag{37}$$

where

$$\Gamma_l \equiv \frac{l(l+1)}{4l^3+6l^2-1} \tag{38}$$

and

$$B_l \equiv \frac{2l+1}{4l^3+6l^2-1}\,. \tag{39}$$

It is convenient to characterize the external flow field $\mathbf{v}^{\infty}(\mathbf{r})$ in a similar manner. We therefore define

$$\mathbf{V}^{\infty}(\mathbf{r}) \equiv \mathbf{v}^{\infty}(R,\theta,\phi), \tag{40}$$

its normal component

$$X^{\infty} \equiv \sum_{l,m} X^{\infty}_{l,m} \mathcal{Y}_{l,m}(\theta,\phi) \equiv \mathbf{V}^{\infty}(R,\theta,\phi)\mathbf{e}_r \tag{41}$$

and its divergence

$$Y^{\infty} \equiv \sum_{l,m} Y^{\infty}_{l,m} \mathcal{Y}_{l,m}(\theta,\phi) \equiv -R\nabla\mathbf{V}^{\infty}. \tag{42}$$

The equation of motion (18) for the quasi-spherical vesicle in this external flow thus becomes

$$\partial_t \mathbf{R}(\theta,\phi) = \mathbf{V}^{\infty}(\theta,\phi) + \mathbf{V}^{ind}(\theta,\phi). \tag{43}$$

The local incompressiblility condition (20) becomes after little algebra

$$\nabla(\mathbf{V}^{\infty} + \mathbf{V}^{ind}) = 0, \tag{44}$$

or

$$Y^{ind} = -Y^{\infty}. \tag{45}$$

Physically, the incompressibility condition fixes the local surface tension which we have already eliminated in favor of Y^{ind}.

We will now set up the equation of motion for the normal deviation, $u(\theta,\phi)$, from the perfect sphere, where θ and ϕ are fixed coordinates in the lab frame. It is important to realize that (43) leads to a tangential motion of the material point labeled by (θ,ϕ). Therefore it is not sufficient to project (43) onto $\mathbf{e}_r$ in order to get the normal shape change in the lab frame. We also have to include an advection term as for planar membranes [29,30]. The equation of motion thus becomes

$$\partial_t u(\theta,\phi) = -(\mathbf{V}^{\infty} + \mathbf{V}^{ind})\nabla u(\theta,\phi) + (X^{\infty} + X^{ind})/R. \tag{46}$$

Note that the advection term can be written in terms of three-dimensional vectors because u carries no r dependence. In general, the advection term will couple the different modes. Further progress becomes possible for the quite wide class of linear external flow.

3.3 Langevin equation for linear flow

Linear external flow is characterized by the form

$$\mathbf{v}^{\infty}(\mathbf{r}) = \mathcal{G}\mathbf{r} \tag{47}$$

where $\mathcal{G}$ is a traceless $\mathbf{r}$-independent matrix. It can be decomposed according to

$$\mathbf{v}^{\infty}(\mathbf{r}) = \mathcal{G}_s\mathbf{r} + \mathbf{\Omega}\times\mathbf{r}, \tag{48}$$

where $\mathbf{\Omega}$ denotes magnitude and direction of the rotational part of the flow and $\mathcal{G}_s$ is a symmetric traceless matrix describing the strain or irrotational component of the flow. The strain component of the external flow is compensated by the corresponding induced flow because of the incompressibility constraint. The advection term thus involves only the rotational component of the external flow and reads

$$-(\mathbf{\Omega}\times\mathbf{r})\nabla u(\theta,\phi) = -i\mathbf{\Omega}(\mathbf{L}/\hbar)u(\theta,\phi), \tag{49}$$

where $\mathbf{L} \equiv \mathbf{r}\times(-i\hbar\nabla)$ is the angular momentum operator. Choosing co-ordinates such that $\mathbf{\Omega} = \Omega\mathbf{e}_z$, we finally obtain from (46) the equation of motion

$$\begin{aligned}\partial_t u_{l,m} = &-i\Omega m\; u_{l,m} - (\kappa/\eta R^3)\Gamma_l E_l\; u_{l,m}\\ &+ (X^{\infty}_{l,m} + B_l Y^{\infty}_{l,m})/R + \zeta_{l,m}\end{aligned} \tag{50}$$

after expanding in spherical harmonics and using (37, 45).

In order to determine the correlations of the thermal noise $\zeta_{l,m}$ we apply this equation of motion to equilibrium [11–13]. Then, there is no external flow, $\Omega = X^{\infty}_{l,m} = Y^{\infty}_{l,m} = 0$. In the long time limit, the dynamical equal time correlations calculated from (50) will reproduce the static correlations only if we choose

$$\langle\zeta_{l,m}(t)\zeta_{l',m'}(t')\rangle = 2T(\Gamma_l/\eta R^3)(-1)^m\delta_{l,l'}\delta_{m,-m'}\delta(t-t'). \tag{51}$$

Here, T is the temperature and Boltzmann's constant is set to 1 throughout.

We keep the noise correlations (51) in the presence of external flow. While this is certainly correct for small flow it is not clear whether and how strong flow modifies these correlations. Note in passing that we have refrained from adding noise to the general equation of motion (18), since the appropriate correlations for these forces are not clear for the same reason. Naively, one would expect them to exhibit long-range spatial correlations given by the Oseen tensor for small flow. Moreover, even in equilibrium, there are subtleties associated both with the incompressibility constraint and measure factors [9].

The stationary value $\bar{u}_{l,m}$ of the amplitude $u_{l,m}(t)$ follow from (50) as

$$\bar{u}_{l,m} = \left(\frac{\eta R^2}{\kappa}\right)\left(\frac{X^\infty_{l,m} + B_l Y^\infty_{l,m}}{\Gamma_l E_l + i\tilde{\Omega} m}\right), \tag{52}$$

where

$$\tilde{\Omega} \equiv \Omega \eta R^3/\kappa. \tag{53}$$

For linear flow, $\bar{u}_{l,m} \neq 0$ only for $l = 2$.

The deviations from this stationary value,

$$\epsilon_{l,m}(t) \equiv u_{l,m}(t) - \bar{u}_{l,m}, \tag{54}$$

obey the equation of motion

$$\partial_t\, \epsilon_{l,m} = -i\Omega m\, \epsilon_{l,m} - (\kappa/\eta R^3)\Gamma_l E_l\, \epsilon_{l,m} + \zeta_{l,m}. \tag{55}$$

This simple equation is easily solved as

$$\epsilon_{l,m}(t) = \exp[(-(\kappa/\eta R^3)\Gamma_l E_l - i\Omega m)t] \times \left(\int_o^t d\tau \exp[((\kappa/\eta R^3)\Gamma_l E_l + i\Omega m)\tau]\zeta_{l,m}(\tau) + \epsilon_{l,m}(0)\right). \tag{56}$$

Starting from an initial value $\epsilon_{l,m}(0)$, the approach to the stationary value (52) thus happens *via* damped oscillations up to thermal fluctuations.

Using the noise correlations (51), the dynamical correlation function in the long time limit become

$$\lim_{t\to\infty} \langle \epsilon_{l,m}(t)\epsilon_{l,-m}(t+\Delta t)\rangle = (-1)^m \exp[(-(\kappa/\eta R^3)\Gamma_l E_l + im\Omega)\Delta t]\frac{T}{\kappa E_l} \tag{57}$$

with the stationary correlations

$$\lim_{t\to\infty} \langle \epsilon_{l,m}(t)\epsilon_{l,-m}(t)\rangle = (-1)^m \frac{T}{\kappa E_l}\cdot \tag{58}$$

3.4 Area constraint and effective tension

As a last step, we have to eliminate the yet unknown Lagrange multiplier or effective tension σ in favor of physical quantities using the area constraint (28). Excess area is stored both in the systematic stationary amplitudes $\bar{u}_{l,m}$ as well as in fluctuations $\epsilon_{l,m}$. For the systematic part, we get

$$\bar{\Delta} \equiv \sum_{l,m} \frac{(l+2)(l-1)}{2}|\bar{u}_{l,m}|^2 = 2\left(\frac{\eta R^2}{\kappa}\right)^2 \sum_{m=-2}^{2}\left(\frac{(X^\infty_{2,m} + B_2 Y^\infty_{2,m})^2}{\Gamma_2^2 E_2^2 + \tilde{\Omega}^2 m^2}\right). \tag{59}$$

The fluctuation contribution $\Delta_{l,m}$ of the mode $\epsilon_{l,m}$ to the excess area is

$$\Delta_{l,m} \equiv \frac{(l+2)(l-1)}{2}|\epsilon_{l,m}|^2 = \frac{(l+2)(l-1)}{2}\frac{T}{\kappa E_l}\cdot \tag{60}$$

The total area constraint becomes

$$\Delta = 2\left(\frac{\eta R^2}{\kappa}\right)^2 \sum_{m=-2}^{2}\left(\frac{(X^\infty_{2,m} + B_2 Y^\infty_{2,m})^2}{\Gamma_2^2 E_2^2(\sigma) + \tilde{\Omega}^2 m^2}\right) + \frac{T}{2\kappa}\sum_{l,m}\frac{(l+2)(l-1)}{E_l(\sigma)}, \tag{61}$$

where we made the σ-dependence of E_l explicit. This master equation determines the unknown effective tension σ as a function of the physical parameters T/κ and those characterizing the external flow. In general, it must be solved numerically.

4 Shear flow

In this section, we specialize the general results of the previous section to the experimentally important case of simple shear flow with shear rate $\dot{\gamma}$.

4.1 Flow parameters

We choose the coordinate system such that the externally imposed simple shear flow reads

$$\mathbf{v}^\infty(\mathbf{r}) = \dot{\gamma} y \mathbf{e}_x = \dot{\gamma}[(y/2)\mathbf{e}_x + (x/2)\mathbf{e}_y)] - (\dot{\gamma}/2)\mathbf{e}_z \times \mathbf{r}. \tag{62}$$

We have separated the rotational component and identify

$$\Omega = -\dot{\gamma}/2 \quad \text{and} \quad \tilde{\Omega} = -\chi/2, \tag{63}$$

where

$$\chi \equiv \dot{\gamma}\eta R^3/\kappa \tag{64}$$

is the dimensional shear rate. The elongational first part of the shear flow can be written as

$$\dot{\gamma}[(y/2)\mathbf{e}_x + (x/2)\mathbf{e}_y)] = \dot{\gamma} r(\sin^2\theta \sin\phi\cos\phi \mathbf{e}_r + (1/2)\sin\theta(\cos^2\phi - \sin^2\phi)\mathbf{e}_\phi + \sin\theta\cos\theta\sin\phi\cos\phi\mathbf{e}_\theta). \tag{65}$$

One thus has

$$X^{\infty} = Y^{\infty} = \dot{\gamma} R \sin^2\theta \sin\phi \cos\phi \tag{66}$$

and identifies the expansion coefficients in spherical harmonics as

$$X^{\infty}_{2,\pm 2} = Y^{\infty}_{2,\pm 2} = \mp i\dot{\gamma} R (2\pi/15)^{1/2} \tag{67}$$

and any other $X^{\infty}_{l,m} = Y^{\infty}_{l,m} = 0$. Note as an aside that $X^{\infty} = Y^{\infty}$ holds for any linear flow.

4.2 Effective tension

We must eliminate the effective tension σ in favor of the area constraint. The external shear flow implies the non-zero stationary amplitudes (52)

$$\bar{u}_{2,\pm 2} = \frac{\mp i\chi(12/11)(2\pi/15)^{1/2}}{\Gamma_2 E_2 \mp i\chi}, \tag{68}$$

where we used $B_2 = 1/11$. According to (59) the systematic part of excess area stored in the modes $\bar{u}_{2,\pm 2}$ contributes

$$\bar{\Delta} = 2(|\bar{u}_{2,2}|^2 + |\bar{u}_{2,-2}|^2) = a_2 \frac{\chi^2}{\Gamma_2^2 E_2^2 + \chi^2} \tag{69}$$

where

$$a_2 \equiv 4(12/11)^2 2\pi/15 \simeq 1.994. \tag{70}$$

The area stored in the fluctuations becomes

$$\frac{T}{2\kappa}\sum_{l,m}\frac{(l+2)(l-1)}{E_l} = \frac{T}{2\kappa}\sum_{l\geq 2}\frac{2l+1}{l(l+1)+\sigma} = \frac{T}{2\kappa}\frac{5}{6+\sigma} + \frac{T}{2\kappa}\sum_{l\geq 3}\frac{2l+1}{l(l+1)+\sigma}\cdot \tag{71}$$

Replacing the last sum by an integral, we obtain the total area constraint in the form

$$a_2\frac{\chi^2}{\Gamma_2^2 E_2^2 + \chi^2} + \frac{T}{2\kappa}\left(\frac{5}{6+\sigma}\right) + \frac{T}{2\kappa}\ln\left(\frac{l_{max}^2+\sigma}{12+\sigma}\right) = \Delta. \tag{72}$$

This equation can easily be solved numerically for $\sigma = \sigma(\chi, T/\kappa, l_{max})$. However, it is more instructive to discuss limiting cases and the general behavior analytically.

For vanishing shear rate, *i.e.*, in equilibrium, the area constraint implies the three regimes [17]):

(i) *Tense regime:* For $\Delta \ll T/2\kappa$, one obtains from (72)

$$\sigma \approx \frac{T}{2\kappa\Delta} l_{max}^2. \tag{73}$$

In this regime, all N modes share the available excess area.

(ii) *Entropic regime:* For $T/2\kappa \ll \Delta \ll (T/\kappa)\ln l_{max}$, the tension depends exponentially on the excess area [18]

$$\sigma \approx l_{max}^2 e^{-2\kappa\Delta/T}. \tag{74}$$

(iii) *Prolate regime:* For $(T/\kappa)\ln l_{max}^2 \ll \Delta \lesssim 1$, most of the excess area is stored in the $(l = 2)$-modes. The tension approaches the limiting value -6 [13,17]:

$$\sigma \approx -6 + \frac{5}{2}\frac{T}{\kappa\Delta}\cdot \tag{75}$$

A vesicles that is relaxed with respect to its volume constraint also belongs to this regime with $\sigma = 0$.

For large shear rate, *i.e.*, formally $\chi \to \infty$, the whole excess area is stored in the $\bar{u}_{2,\pm 2}$ deformation. From

$$\bar{\Delta} = 2(|\bar{u}_{2,2}|^2 + |\bar{u}_{2,-2}|^2) = a_2\frac{\chi^2}{\Gamma_2^2 E_2^2 + \chi^2} = \Delta, \tag{76}$$

one obtains

$$\Gamma_2 E_2 = \chi\left(\frac{a_2 - \Delta}{\Delta}\right)^{1/2} \tag{77}$$

and hence, with $E_2 \approx 4\sigma$ and $\Gamma_2 = 6/55$, the limiting behavior

$$\sigma \approx (55/24)a_2^{1/2}\chi/\Delta^{1/2} \simeq 3.24\chi/\Delta^{1/2}\cdot \tag{78}$$

Thus, for large shear rate, the tension increases linearly with shear rate with a prefactor that depends as an inverse square root on the excess area.

The cross-over between the equilibrium tension and this shear rate dominated saturation regime happens at χ_c. The scaling behavior of the cross-over shear rate χ_c can be obtained by defining χ_c as the value for which half of the excess area is stored in the systematic contribution $\bar{\Delta}$. This leads to the expression

$$\chi_c \sim \begin{cases} l_{max}^2 T/\kappa\Delta, & \text{tense,} \\ l_{max}^2 e^{-2\kappa\Delta/T}, & \text{entropic,} \\ T/\kappa\Delta, & \text{prolate.} \end{cases} \tag{79}$$

4.3 Shape parameters

Knowing the value of the effective tension as a function of the excess area, we can determine the characteristic parameters of the shear rate dependent shape of a quasi-spherical vesicle.

The deviation of the mean contour from a circle in the plane $z = 0$, *i.e.*, $\theta = \pi/2$, becomes

$$u(\pi/2,\phi) = 2\mathcal{R}e\{u_{2,2}\mathcal{Y}_{2,2}(\pi/2,\phi)\} = \frac{6}{11}\frac{\chi}{(\Gamma_2^2 E_2^2 + \chi^2)^{1/2}}\cos 2(\phi - \phi_0), \tag{80}$$

with the mean inclination angle

$$\phi_0 = \frac{1}{2}\arctan\frac{\Gamma_2 E_2}{\chi}\cdot \tag{81}$$

Thus, the mean inclination angle is $\phi_0 = \pi/4$ for small shear flow. For large shear rate, the limiting behavior

$$\phi_0 \approx \frac{1}{2}\arctan\left(\frac{a_2-\Delta}{\Delta}\right)^{1/2} \approx \pi/4 - \frac{\Delta^{1/2}}{2a_2^{1/2}} \quad (82)$$

follows from (77). Thus, the more excess area is available, the smaller the inclination angle. The cross-over between the two limiting cases happens at the cross-over shear rate χ_c given in (79).

Experimentally, the ellipticity of such a contour is often measured by the deformation parameter

$$D \equiv \frac{L-B}{L+B} = \frac{6}{11}\frac{\chi}{(\Gamma_2^2 E_2^2+\chi^2)^{1/2}}, \quad (83)$$

where $2L$ and $2B$ are the long and short axes, resp., of the contour. For small shear rate χ, the equilibrium scaling of the tension (73–75) in the three regimes implies the following linear behavior of the deformation parameter D on χ as

$$D \approx \frac{6}{11}\frac{\chi}{\Gamma_2 E_2} = \frac{6}{11}\frac{\chi}{(24/55)(6+\sigma)} \approx \frac{5}{4}\chi\cdot\begin{cases}\dfrac{2\kappa\Delta}{Tl_{max}^2}, & \text{tense,}\\ e^{2\kappa\Delta/T}/l_{max}^2, & \text{entropic,}\\ (2\kappa\Delta/5T), & \text{prolate.}\end{cases} \quad (84)$$

With increasing shear rate, deviations from this linear behavior become important at the cross-over χ_c (79). For $\chi \gg \chi_c$, the deformation saturates at the value

$$D \approx \sqrt{15\Delta/32\pi}, \quad (85)$$

obtained by combining (83) with (69).

4.4 Comparison to numerical work

For large shear rate, fluctuations become irrelevant. In this regime, we can make contact with previous numerical work where the continuum equation of motions (11, 20) were solved on a triangulated mesh [5]. Of course, one should compare results only for small excess area where deviations from the spherical shape are small. The two quantities that can be compared within the two approaches are the asymptotic dependence of the inclination angle and the effective tension on the excess area. In the previous work, reduced volume

$$v \equiv \frac{V}{(4\pi/3)(A/4\pi)^{3/2}} = (1+\Delta/4\pi)^{-3/2} \quad (86)$$

was used instead of the excess area. This definition implies

$$\Delta = 4\pi(v^{-2/3}-1). \quad (87)$$

Hence the present theory predicts for large shear rate and $1-v \ll 1$ the limiting behavior

$$\phi_0 \approx \frac{\pi}{4} - \left(\frac{2\pi(1-v)}{3a_2}\right)^{1/2} \simeq \frac{\pi}{4} - 1.025(1-v)^{1/2}. \quad (88)$$

This square-root scaling is in quantitative agreement with the previous numerical solution [5]. The numerical prefactor within the present approach is about 10% larger than the one extracted from Figure 2 of reference [5]. Likewise, the scaling of the effective tension (78) agrees with the numerical data [41] with a difference in prefactor of about 30%.

The origin of the numerical difference in prefactors presumably lies in neglecting higher order couplings between deformation and external flow in the equation of motion (50). These effects would lead to non-diagonal and nonlinear terms in the right hand side of (50) which would spoil the solvability. It is therefore gratifying to know that at least for the inclination angle of the tanktreading state, the present treatment is quite reliable for $v \geq 0.8$ at any shear rate.

4.5 Comparison to experiment

There seems to be only one published experiment on quasi-spherical vesicles in shear flow [8]. In this work, two different types of behavior have been reported. (i) Initially spherical vesicles rotate at low shear rate and deform at higher shear rates into a tanktreading state with an inclination angle close to $\pi/4$. (ii) Initially nonspherical vesicles undergo a periodic flipping motion at small shear rate. This flipping motion shows oscillations at larger shear rate. Only for sufficiently high shear rate, stationary behavior with tanktreading is observed.

The first class of behavior fits well into the theoretical description developed here. The experimentally found $D(\dot\gamma)$ curve shows linear behavior at small $\dot\gamma$ and seems to saturate at larger shear rate. de Haas *et al.* [8] analyzed this curve using a theoretical approach which may look superficially similar to the one developed here. For future reference, we discuss briefly the differences between these two approaches. De Haas *et al.* use

$$\dot\gamma = \frac{4\Sigma D}{5R\eta}\exp\frac{64\pi\kappa}{15T}D^2, \quad (89)$$

in our notation. The corresponding expression within our approach derived from (83) reads

$$\dot\gamma = \frac{4D\Sigma}{5\eta R}(1+\frac{6\kappa}{\Sigma R^2})\left(1-\left(\frac{11D}{6}\right)^2\right)^{-1/2} \quad (90)$$

in dimensionalized units. The κ term in the first parenthesis arises from keeping bending energy in the force balance. De Haas *et al.* deliberately ignore bending as being irrelevant for large tension which is true in the tense and upper part of the entropic regime. Then both expression yield the same linear behavior at small D. The non-linear regime, however, is markedly different with no exponential term in our theory. Note also that, within our approach, Σ still carries an implicit $\dot{\gamma}$ dependence.

The second type of reported behavior cannot so easily be reconciled with the present theory which predicts a stationary tanktreading shape for all values of shear rates and excess area. However, the theory allows for a transient damped oscillatory approach to this stationary state. In the prolate regime, the relaxation time

$$\tau_2 \equiv \eta R^3/\kappa \Gamma_2 E_2 \tag{91}$$

can become large for large enough excess area because then formally $E_2 \to 0$ as $\Delta \to \infty$. For large enough shear rate, the product of oscillation frequency $\dot{\gamma}$ with the relaxation time τ_2 can become large. In this case, many oscillations can occur before the stationary state is reached. Thus, after turning on shear flow or changing the shear rate one expects transient flipping motion and oscillatory behavior. Still, the present theory predicts finally a settling into a stationary tanktreading state.

5 Discussion

For a fluid membrane in an external hydrodynamic field, we derived a deterministic equation of motion taking into account both viscous dissipation in the surrounding liquid and local incompressibility of the membrane. For quasi-spherical vesicles in linear external flow, this equation decouples in an expansion in spherical harmonics. The area constraint is implemented by an effective tension that depends on shear rate. Adding noise with correlations adapted from equilibrium, we obtain a Langevin-type equation. Its solution for shear flow yields transient oscillations which settle into a stationary tanktreading state. The inclination angle and the deformation parameter are both calculated as a function of shear rate and excess area. For small shear rate, the inclination angle becomes $\pi/4$. With increasing shear rate it decreases the stronger the larger the excess area is. The deformation parameter increases linearly with shear rate with a slope depending on excess area.

This work is complementary to our previous study where we solved the deterministic equations numerically for arbitrary excess area ignoring fluctuations [5]. Taken together, the two approaches yield a fairly complete picture for the behavior of fluid vesicles in all phase space. There remains only one explored region. For an excess area too large for the quasi-spherical approximation to apply and a shear rate too small for the numerics to handle, neither approach is well-suited. In this regime, one would naively expect that small shear leads to small distortions around the fluctuating equilibrium shape. However, this remains to be checked by an explicit calculation.

A relatively straightforward extension of the present theory would be the inclusion of further dissipative mechanisms such as inter-monolayer drag and shear within each monolayer. While both types will certainly affect the transient behavior, it is less likely that monolayer shear viscosity changes the stationary results for quasi-spherical vesicles since the rotational component of the flow does not lead to shear within the layer. Likewise, one could easily allow for different viscosities inside and outside the vesicle.

The analytical progress possible within the quasi-spherical approximation comes with two intrinsic weaknesses which cannot easily be circumvented. First, in the prolate regime, this approximation gives only a rough representation of the stationary ellipsoidal mode since it does not produce a non-zero $u_{2,m}$ without shear. Second, this approximation neglects higher order couplings between deformation and flow field which can become relevant at larger shear rates. However, the relatively good agreement between the present theory and the previous numerical work shows that this effect is certainly not dominant for $v \gtrsim 0.8$.

The present study should encourage further experimental work. By measuring the inclination angle and the deformation parameter of a single vesicle over the whole range of shear rates and fitting those against theory using the full equation (72), one should be able to extract fairly precise data on excess area and bending rigidity especially when combined with a variation of temperature [42].

Concerning the experimentally reported oscillatory or flipping motion, it seems too early for a final assessment. While it could be that the above mentioned effects mask such a motion within the theory, there are two more indications for doubting that flipping motion is genuine for non-spherical vesicles. First, our numerical work [5] which did not suffer from the spherical approximation, did not show flipping motion for at least $\chi \gtrsim 0.1$. Secondly, contrary to the reference given by de Haas *et al.* [8], previous work on tanktreading of red-blood-cells [24] did not predict flipping motion if the viscosities inside and outside are the same. Only if the inner viscosity is significantly larger that the outer, a transition to flipping motion occurs.

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Appendix: Adaptation of Lamb's solution

Lamb's solution tells us the velocity field $\mathbf{v}(\mathbf{r})$ and pressure field $p(\mathbf{r})$ in all space if the velocity field is specified on a sphere with radius R. We adapt the presentation of this classical problem given in reference [33] whose notation we follow. Rather than characterizing $\mathbf{v}(R, \theta, \phi)$ by its three components, Lamb's solution uses the three quantities

$$X(\theta, \phi) \equiv \mathbf{V}(\mathbf{r})\mathbf{e}_r, \tag{A.1}$$

$$Y(\theta, \phi) = -R\ \nabla \mathbf{V}(\mathbf{r}), \tag{A.2}$$

and

$$Z(\theta, \phi) = R\ \mathbf{e}_r(\nabla \times \mathbf{V}(\mathbf{r})), \tag{A.3}$$

derived from the field $\mathbf{V}(\mathbf{r}) \equiv \mathbf{v}(R, \theta, \phi)$. Note that $\mathbf{V}(\mathbf{r})$ is independent of r. Since the rotational component Z cannot be excited by either bending moments or inhomogeneities in the surface tension, we can ignore it in the following.

Given these boundary values of the velocity, the total velocity field for $r < R$ becomes

$$\mathbf{v}^{in}(\mathbf{r}) = \sum_{l\geq 2} \left(\nabla \Phi_l^{in}(\mathbf{r}) + \frac{l+3}{2\eta(l+1)(2l+3)} r^2 \nabla p_l^{in}(\mathbf{r}) - \frac{l}{\eta(l+1)(2l+3)} \mathbf{r} p_l^{in}(\mathbf{r}) \right) \quad \text{(A.4)}$$

where

$$\Phi_l^{in}(\mathbf{r}) = \sum_{m=-l}^{l} \Phi_{l,m}^{in} \mathcal{Y}_{l,m}(\theta, \phi)(r/R)^l \quad \text{(A.5)}$$

and

$$p_l^{in}(\mathbf{r}) = \sum_{m=-l}^{l} p_{l,m}^{in} \mathcal{Y}_{l,m}(\theta, \phi)(r/R)^l. \quad \text{(A.6)}$$

The expansion coefficients $\Phi_{l,m}^{in}, p_{l,m}^{in}$ are determined by the boundary conditions as

$$\Phi_{l,m}^{in}(\mathbf{r}) = \frac{R}{2l}[(l+1)X_{l,m} - Y_{l,m}] \quad \text{(A.7)}$$

and

$$p_{l,m}^{in}(\mathbf{r}) = \frac{\eta(2l+3)}{lR}[Y_{l,m} - (l-1)X_{l,m}]. \quad \text{(A.8)}$$

For $r > R$, the velocity field can be obtained formally by replacing l by $-l-1$ in these expressions. Explicitly, it reads

$$\mathbf{v}^{out}(\mathbf{r}) = \sum_{l\geq 1} \left(\nabla \Phi_l^{out}(\mathbf{r}) + \frac{-l+2}{2\eta(-l)(-2l+1)} r^2 \nabla p_l^{out}(\mathbf{r}) - \frac{-l-1}{\eta(-l)(-2l+1)} \mathbf{r} p_l^{out}(\mathbf{r}) \right) \quad \text{(A.9)}$$

where

$$\Phi_l^{out}(\mathbf{r}) = \sum_{m=-l}^{l} \Phi_{l,m}^{out} \mathcal{Y}_{l,m}(\theta, \phi)(r/R)^{(-l-1)} \quad \text{(A.10)}$$

and

$$p_l^{out}(\mathbf{r}) = \sum_{m=-l}^{l} p_{l,m}^{out} \mathcal{Y}_{l,m}(\theta, \phi)(r/R)^{(-l-1)} \quad \text{(A.11)}$$

with

$$\Phi_{l,m}^{out}(\mathbf{r}) = \frac{R}{2(l+1)}[lX_{l,m} + Y_{l,m}] \quad \text{(A.12)}$$

and

$$p_{l,m}^{out}(\mathbf{r}) = \frac{\eta(2l-1)}{(l+1)R}[Y_{l,m} + (l+2)X_{l,m}]. \quad \text{(A.13)}$$

This velocity field leads to a stress vector acting across the surface of a sphere of radius $r = R$ as

$$\mathbf{T}_R \equiv T_R \mathbf{e}_r + \mathbf{T}_t \equiv \left(-\mathbf{e}_r p + \eta \left(\frac{\partial \mathbf{v}}{\partial r} - \frac{\mathbf{v}}{r} \right) + \frac{\eta}{r} \nabla(\mathbf{r}\mathbf{v}) \right)_{|r=R}. \quad \text{(A.14)}$$

The normal component of this stress vector at $r = R$ reads for the inner solution

$$T_R^{in} = \sum_{l,m} (2(\eta/R^2)(l-1)l\Phi_{l,m}^{in} + (la_l^{in} - b_l^{in})p_{l,m}^{in})\mathcal{Y}_{l,m}(\theta, \phi), \quad \text{(A.15)}$$

where

$$a_l^{in} \equiv \frac{l(l+2)}{(l+1)(2l+3)} \quad \text{(A.16)}$$

and

$$b_l^{in} \equiv \frac{2l^2 + 4l + 3}{(l+1)(2l+3)}. \quad \text{(A.17)}$$

Likewise, the normal component of the stress vector at $r = R$ reads for the outer solution

$$T_R^{out} = \sum_{l,m} ([2(\eta/R^2)(l+2)(l+1)\Phi_{l,m}^{out} - [(l+1)a_l^{out} + b_l^{out})p_{l,m}^{out}])\mathcal{Y}_{l,m}(\theta, \phi) \quad \text{(A.18)}$$

where

$$a_l^{out} \equiv \frac{l^2 - 1}{l(2l-1)} \quad \text{(A.19)}$$

and

$$b_l^{out} \equiv \frac{2l^2 + 1}{l(2l-1)}. \quad \text{(A.20)}$$

The tangential part at $r = R$ reads for the inner solution

$$\mathbf{T}_t^{in} = \sum \left(2(\eta/R^2)(l-1)\Phi_{l,m}^{in} + a_l^{in} p_{l,m}^{in} \right) \nabla^S \mathcal{Y}_{l,m}(\theta, \phi), \quad \text{(A.21)}$$

and for the outer solution

$$\mathbf{T}_t^{out} = \sum \left(-2(\eta/R^2)(l+2)\Phi_{l,m}^{out} + a_l^{out} p_{l,m}^{out} \right) \times \nabla^S \mathcal{Y}_{l,m}(\theta, \phi), \quad \text{(A.22)}$$

where

$$\nabla^S \equiv \frac{1}{\sin\theta} \mathbf{e}_\phi \partial_\phi + \mathbf{e}_\theta \partial_\theta. \quad \text{(A.23)}$$

Accommodation of these well-known properties of Lamb's solution to our problem requires to balance the stress discontinuities at the vesicle surface with those exerted by the vesicle's bending moments and its inhomogeneous surface tension. The key approximation involved for quasispherical vesicles is to assume that all stresses act on a sphere of radius R rather than on the time-dependent vesicle's shape. This approximation can be controlled for small external flow and small excess area. However, in the absence of a workable alternative, we will use it for all flow strengths. The stress balance thus reads

$$\mathbf{T}_R^{out} = \mathbf{T}_R^{in} + \frac{1}{\sqrt{g}}\frac{\delta F}{\delta \mathbf{R}}. \tag{A.24}$$

The stress exerted by the vesicle follows from (11) in an expansion in spherical harmonics as

$$\frac{1}{\sqrt{g}}\frac{\delta F}{\delta \mathbf{R}} = (\kappa/R^3)\sum((E_l u_{l,m} + 2\sigma_{l,m})\mathcal{Y}_{l,m}(\theta,\phi)\mathbf{e}_r - \sigma_{l,m}\nabla^S\mathcal{Y}_{l,m}(\theta,\phi)). \tag{A.25}$$

Evaluating the tangential balance leads to

$$\begin{aligned}(\kappa/\eta R^2)\sigma_{l,m} &= \left(\frac{l-1}{l}\right)[(l+1)X_{l,m} - Y_{l,m}] \\ &+ \left(\frac{l+2}{l+1}\right)[lX_{l,m} + Y_{l,m}] + a_l^{in}\left(\frac{2l+3}{l}\right) \\ &\times [Y_{l,m} - (l-1)X_{l,m}] - a_l^{out}\left(\frac{2l-1}{l+1}\right) \\ &\times [Y_{l,m} + (l+2)X_{l,m}] \\ &= \frac{2l+1}{l(l+1)}(X_{l,m} + 2Y_{l,m}).\end{aligned} \tag{A.26}$$

Evaluating the normal balance, using this result for $\sigma_{l,m}$, and setting $X_{l,m} = X_{l,m}^{ind}$ and $Y_{l,m} = Y_{l,m}^{ind}$ leads to the relations (37–39) quoted in the main part.

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